Recent developments and applications of the finite-amplitude method for nuclear collective excitations

Nobuo Hinohara

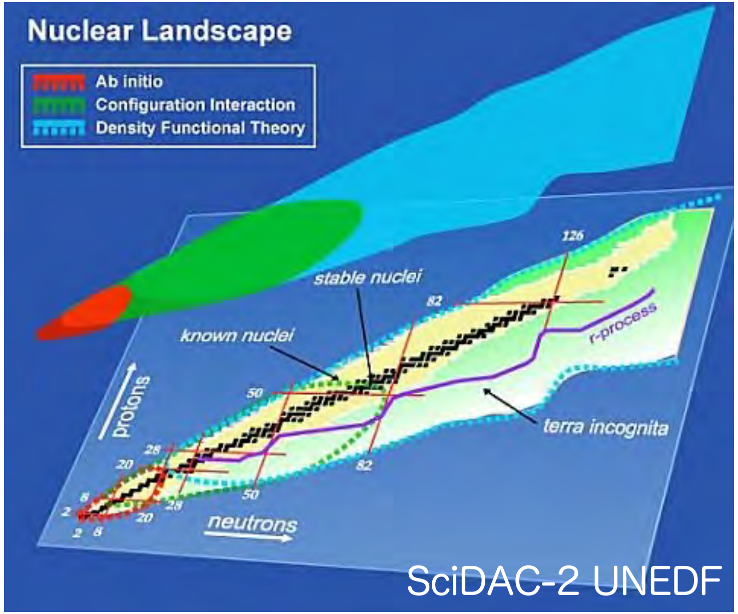
Center for Computational Sciences, University of Tsukuba, Japan

筑波大学
University of Tsukuba

筑波大学
計算科学研究センター
Center for Computational Sciences

Nuclear density functional theory (DFT)

A microscopic theory that can compute from all unstable nuclei to neutron star



Hartree-Fock-Bogoliubov equation for nuclear ground state

iterative eigenvalue problem (non-linear eq.)

$$\begin{pmatrix} h[\rho,\tilde{\rho}]-\lambda & \Delta[\rho,\tilde{\rho}] \\ -\Delta^*[\rho,\tilde{\rho}] & -h^*[\rho,\tilde{\rho}]+\lambda \end{pmatrix} \begin{pmatrix} U_k \\ V_k \end{pmatrix} = E_k \begin{pmatrix} U_k \\ V_k \end{pmatrix}$$

matrix elements (potential)

$$h[\rho,\tilde{\rho}] = \frac{\delta E}{\delta \rho} \qquad \Delta[\rho,\tilde{\rho}] = \frac{\partial E}{\partial \tilde{\rho}^*}$$

quasiparticle wave functions

densities

$$U, V \rightarrow \rho, \tilde{\rho}$$

Quasiparticle random-phase approximation (QRPA)

excited states (including beta decay) and dynamics
eigenvalue problem of large dimension (~10^6)

- ◻ Input of the nuclear DFT: energy density functional (EDF, E[ρ])
- ◻ EDF is determined phenomenologically (not directly derived from the nuclear force)
- ◻ EDF: kinetic energy + particle-hole energy + pairing energy + Coulomb

Quasiparticle random-phase approximation (QRPA)

QRPA: Microscopic theory for excited states of nuclei based on the nuclear DFT

QRPA equation (non-Hermitian eigenvalue problem)

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} X^i \\ Y^i \end{pmatrix} = \Omega_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^i \\ Y^i \end{pmatrix}$$

A: Hermitian matrix
B: symmetric matrix
Ω: excitation energy
(X,Y): wave functions

dimension: 10^5-10^6 full diagonalization is computationally demanding
(computation and storage of A and B matrices, diagonalization..)

Various physics applications:
giant resonances – nuclear matter properties
low-energy vibrational collective excitation
beta decay (eigenvalues below Q value)
double-beta decay (contribution with a ~$1/|\Omega|$ factor)
fission inertia (inertia for collective motion)
moment of inertia/ pairing rotational inertia

Solutions of QRPA:

basis reduction: truncation in the two-quasiparticle space (standard)
Lanczos method (Johnson, et al. Comp. Phys. Commun. **120**, 155 (1999))
iterative Arnoldi method (Toivanen et al. Phys. Rev. C **81**, 034312 (2010))
iteration: finite-amplitude method (Nakatsukasa et al., Phys. Rev. C **76**, 024318 (2007))

Finite amplitude method

Another formulation of the QRPA

Nakatsukasa, Inakura, Yabana, Phys. Rev. C **76**, 024318 (2007)
Avogadro and Nakatsukasa, Phys. Rev. C **84**, 014314 (2011)

A weak time-dependent external one-body field is applied

$$\hat{F}(t) = \eta\left(\hat{F}e^{-i\omega t} + \hat{F}^\dagger e^{i\omega t}\right)$$

F: one-body operator such as isovector dipole
η: small number
ω: complex frequency of the external field (parameter)

$$\hat{F} = \sum_{\mu<\nu} F^{20}_{\mu\nu}\alpha^\dagger_\mu\alpha^\dagger_\nu + F^{02}_{\mu\nu}\alpha_\nu\alpha_\mu$$

quasiparticles oscillate with the same time dependence (time-dependent HFB)

$$\alpha^\dagger_\mu(t) = \sum_k \left[U_{k\mu}(t)c^\dagger_k + V_{k\mu}(t)c_k\right]$$

infinitesimal displacement from the HFB quasiparticles

$$\alpha_\mu(t) = \{\alpha_\mu + \delta\alpha_\mu(t)\}e^{iE_\mu t}$$
$$\delta\alpha_\mu(t) = \eta\sum_\nu \alpha^\dagger_\nu\left[X_{\nu\mu}(\omega)e^{-i\omega t} + Y^*_{\nu\mu}(\omega)e^{i\omega t}\right]$$

mean-field Hamiltonian oscillate (as the quasiparticles oscillate)

$$i\frac{\partial\alpha_\mu(t)}{\partial t} = [\hat{H}(t) + \hat{F}(t), \alpha_\mu(t)]$$

$$\hat{H}(t) = \hat{H}_0 + \eta\left[\delta\hat{H}(\omega)e^{-i\omega t} + \delta\hat{H}^\dagger(\omega)e^{i\omega t}\right]$$
$$\delta\hat{H}(\omega) = \sum_{\mu<\nu}\left[\delta H^{20}_{\mu\nu}(\omega)\alpha^\dagger_\mu\alpha^\dagger_\nu + \delta H^{02}_{\mu\nu}(\omega)\alpha_\nu\alpha_\mu\right]$$

FAM equation: first-order terms in η in TDHFB equation

$$\left[\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} - \omega\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = -\begin{pmatrix} F^{20} \\ F^{02} \end{pmatrix}$$

$$\delta H^{20}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'}[A_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}]X_{\mu'\nu'}(\omega) + B_{\mu\nu\mu'\nu'}Y_{\mu'\nu'}(\omega)$$
$$\delta H^{02}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'}[A^*_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}]Y_{\mu'\nu'}(\omega) + B^*_{\mu\nu\mu'\nu'}X_{\mu'\nu'}(\omega)$$

Note: X(ω)、Y(ω) are not the QRPA solutions (eigenvectors)
QRPA is obtained as a small-amplitude limit of the TDHFB (η<<1)

Induced field

quasiparticle oscillation (X,Y) → induced mean-field Hamiltonian (δH^{20}, δH^{02})

$$\delta H^{20}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} [A_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}]X_{\mu'\nu'}(\omega) + B_{\mu\nu\mu'\nu'}Y_{\mu'\nu'}(\omega)$$

$$\delta H^{02}_{\mu\nu}(\omega) = \sum_{\mu'<\nu'} [A^*_{\mu\nu\mu'\nu'} - (E_\mu + E_\nu)\delta_{\mu\mu'}\delta_{\nu\nu'}]Y_{\mu'\nu'}(\omega) + B^*_{\mu\nu\mu'\nu'}X_{\mu'\nu'}(\omega)$$

X, Y : displacement of the quasiparticles

A,B matrices: second functional derivative of EDF (large-dimensional matrices)

In the linear response calculation, A and B matrices are not necessary.
One-body displacement AX+BY and A*Y+B*X (vectors) are necessary.

displacement of the densities ($\delta\rho$ and $\delta\kappa$)

$$\delta\rho(\omega) = UX(\omega)V^T + V^*Y^T(\omega)U^\dagger$$
$$\delta\kappa^{(+)}(\omega) = UX(\omega)U^T + V^*Y^T(\omega)V^\dagger$$
$$\delta\kappa^{(-)}(\omega) = V^*X^\dagger(\omega)V^\dagger + UY^*(\omega)U^T$$

→

displacement of the one-body potential (δh and $\delta\Delta$)

$$\delta h(\omega) = (h[\rho + \eta\delta\rho] - h[\rho])/\eta$$

$$\delta\Delta^{(\pm)}(\omega) = (\Delta[\rho + \eta\delta\rho, \kappa + \eta\delta\kappa^{(\pm)}, \kappa^* + \eta\delta\kappa^{(\mp)}] - \Delta[\rho, \kappa, \kappa^*])\eta$$

→

quasiparticle representation

$$\delta H^{20}(\omega) = U^\dagger\delta h(\omega)V^* - V^\dagger\delta\Delta^{(-)*}(\omega)V^* + U^\dagger\delta\Delta^{(+)}(\omega)U^* - V^\dagger\delta h^T(\omega)U^*$$

$$\delta H^{02}(\omega) = -V^T\delta h(\omega)U + U^T\delta\Delta^{(-)*}(\omega)U - V^T\delta\Delta^{(+)}(\omega)V + U^T\delta h^T(\omega)V$$

calculation of A and B matrices is avoided in the FAM!

Non-axial giant resonances

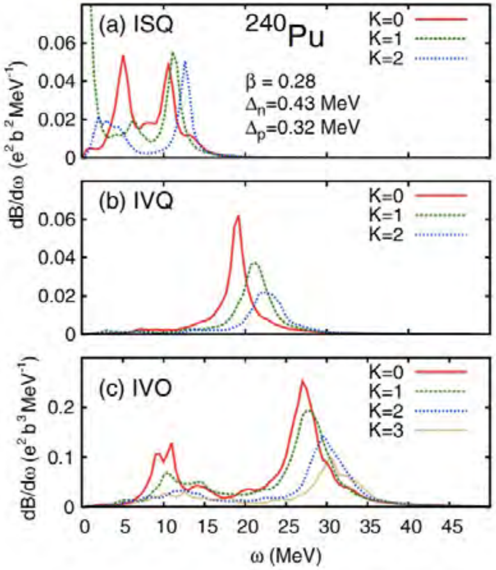
strength function $S(\hat{F},\omega) = \sum_{\mu<\nu} F^{20}_{\mu\nu} X_{\mu\nu}(\omega) + F^{02}_{\mu\nu} Y_{\mu\nu}(\omega)$

Kortelainen, NH, Nazarewicz, Phys. Rev. C **91**, 051302(R) (2015)
Oishi, Kortelainen, NH, Phys. Rev. C **93**, 034329 (2016)

FAM with complex ω+iγ provides strength distribution

FAM based on axial HFB (HFBTHO) (HFB: axial, FAM can be non-axial)

quadrupole and octupole giant resonances in ^{240}Pu



octupole density oscillation in ^{240}Pu

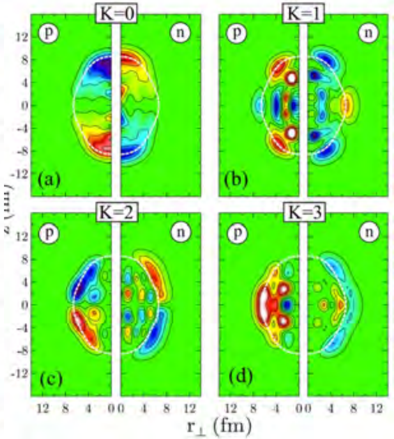
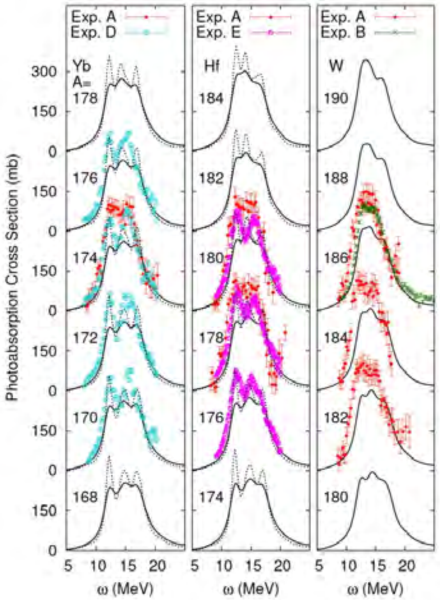
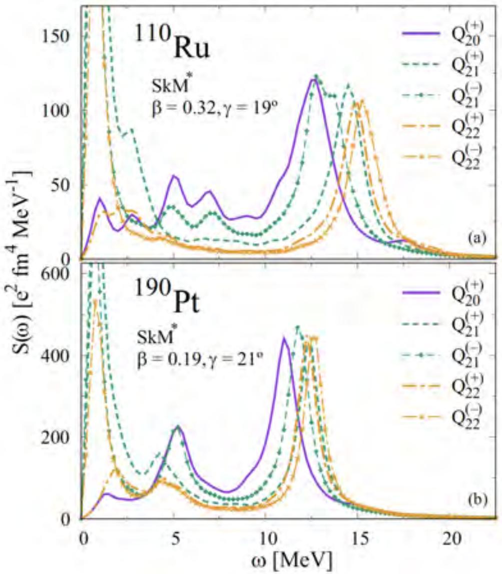


photo absorption cross section in rare-earth region



FAM based on 3D HFB quadrupole giant resonances



Washiyama and Nakatsukasa Phys. Rev. C **96**, 014304(R) (2017)

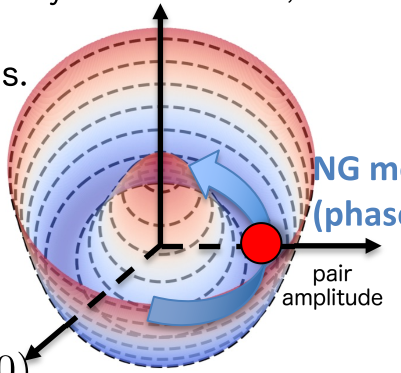
DRHBc-FAM: Talk on Wednesday by Xiaofei Jiang (Peking U.)

Pairing rotation and moment of inertia

NH, Phys. Rev. C **92**, 034321 (2015)
NH and Nazarewicz, Phys. Rev. Lett. **116**,152502 (2016)

NG mode

Nambu-Goldstone (NG) mode appears as a solution of self-consistent QRPA when mean field (DFT) breaks continuous symmetries which the original EDF has.



Pairing rotation

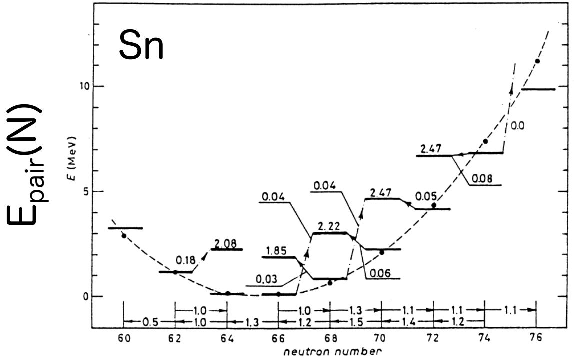
NG mode related to the gauge symmetry breaking due to pairing correlation

$$E_{\text{pair}}(N) = \frac{1}{2\mathcal{J}_{nn}(N_0)}(N - N_0)^2$$

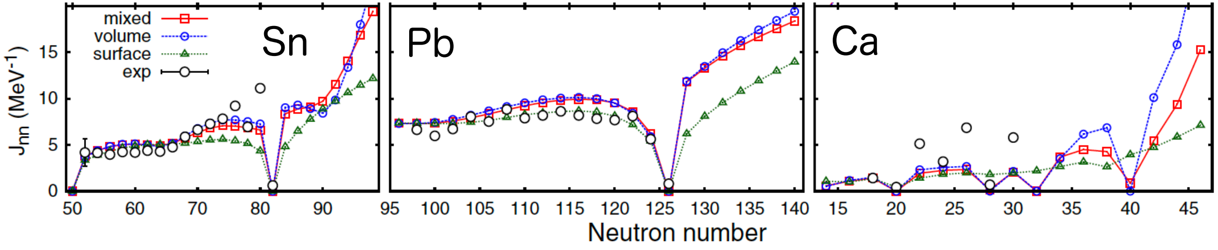
Thouless-Valatin Inertia

$$\mathcal{J}_{nn} = 2N^{20}(A+B)^{-1}N^{20} = -S(\hat{N}_n, \omega = 0)$$

inverse of shell-gap indicator $\mathcal{J}_{nn}^{-1}(N,Z) = \frac{1}{4}\delta_{2n}(N,Z) = \frac{1}{4}[S_{2n}(N,Z) - S_{2n}(N+2,Z)] = \frac{1}{4}[E(N+2,Z) - 2E(N,Z) + E(N-2,Z)]$



Brink and Broglia “Nuclear Superfluidity”



pairing rotational MOI as a function of (N,Z):
indicator of the nuclear pairing

Pairing and quadrupole collectivity

NH and Nazarewicz, Phys. Rev. Lett. **116**, 152502 (2016)

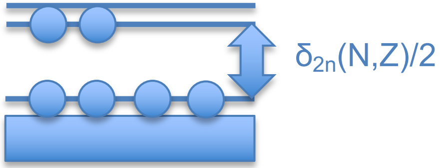
Similarity

pairing collectivity

shell gap indicator (ΔS_{2n})

magic nuclei

signature of magic number



superconducting nuclei

excitation of NG mode

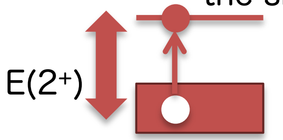
$$\mathcal{J}_{\rm nn}^{-1}(N) = \frac{1}{4}\delta_{\rm 2n}(N)$$

magnitude of pairing collectivity
(gauge symmetry breaking)

$E(2_1^+)$

doubly magic nuclei

ph excitation across the shell gap



deformed nuclei

excitation of NG mode

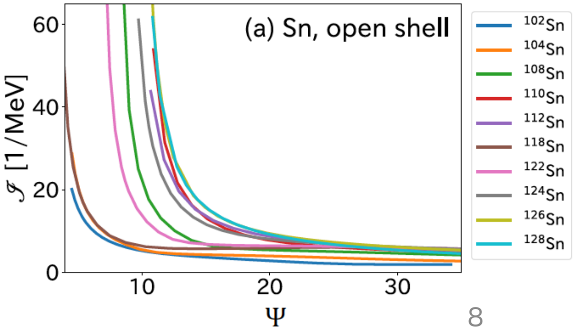
$$E(2_1^+) \sim \frac{3}{\mathcal{J}_{\rm rot}}$$

magnitude of quadrupole collectivity
(deformation)

quadrupole collectivity

Difference

The rotational MoI increases with the order parameter (deformation).
The pairing rotational MoI decreases with the order parameter
(pair amplitude/pairing gap).



C. Ruike, NH, T. Nakatsukasa, arXiv:2504.11908

Contour integration of FAM amplitudes

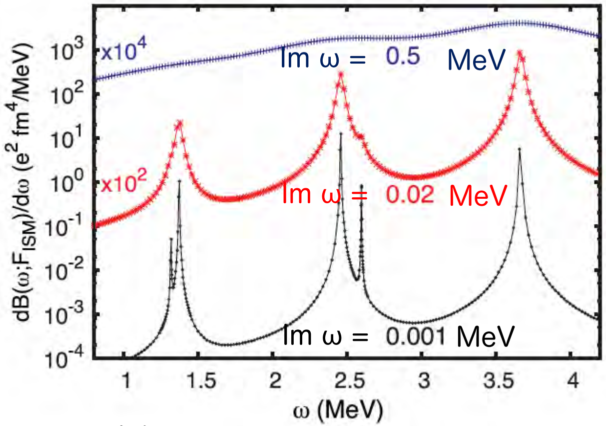
NH, Kortelainen, Nazarewicz, Phys. Rev. C **87**, 064309 (2013)

FAM X and Y amplitudes are not QRPA eigenvectors

Relation between QRPA XY and FAM XY

$$X_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{X^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{Y^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$

$$Y_{\mu\nu}(\omega) = -\sum_i \left\{ \frac{Y^i_{\mu\nu}\langle i|\hat{F}|0\rangle}{\Omega_i - \omega} + \frac{X^{i*}_{\mu\nu}\langle 0|\hat{F}|i\rangle}{\Omega_i + \omega} \right\}$$



FAM XY have first-order poles at QRPA energies with QRPA XY as residues

Kortelainen, NH, Nazarewicz, Phys. Rev. C **92**, 051302(R)(2015)

$$\frac{1}{2\pi i}\oint_{C_i} X_{\mu\nu}(\omega) d\omega = e^{i\theta}|\langle i|\hat{F}|0\rangle| X^i_{\mu\nu}$$

TABLE I. Lowest octupole QRPA modes in ^{154}Sm predicted in our deformed FAM calculations. Shown are the energy ω_1; the IVO transition strength $|\langle 0|f^{\mathrm{IV},+}_{L=3,K}|1\rangle|^2$; and the corresponding $B(E3)$ value. The transition probabilities were computed through the QRPA amplitudes (referred to as FAM-C in Ref. [27]).

K	ω_1 (MeV)	$\lvert\langle 0\lvert f^{\mathrm{IV},+}_{L=3,K}\rvert 1\rangle\rvert^2$ (e^2fm^6)	$B(E3)$ (W.u.)
0	0.80	1.57	8.02
1	1.12	26.20	2.75
2	2.40	0.73	2.36
3	2.49	0.47	0.04

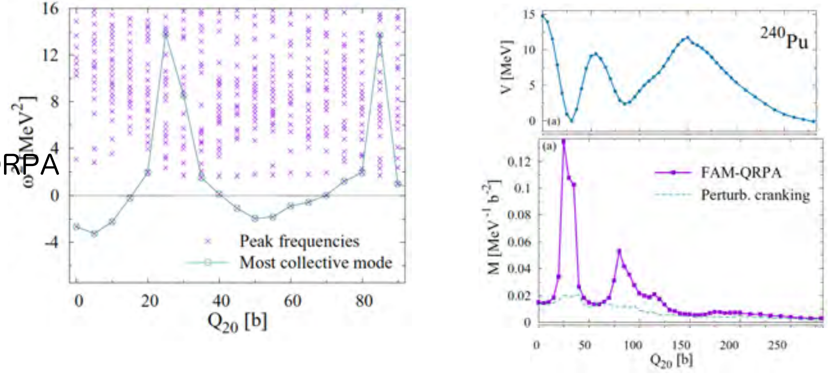
low-energy collective excitation

- contour integration around one peak derives QRPA XY (and Ω)
- numerous FAM calculations for searching low-energy peaks
- applications to low-lying states, fission and vibrational inertia

Application of contour integration FAM

Fission inertia

Washiyama, NH, Nakatsukasa, Phys. Rev. C **103**, 014306 (2021)



local QRPA inertia NH et al., Phys. Rev. C **82**, 064313 (2010)

$$\mathcal{M}(q_{20}) = \frac{dq}{dq_{20}}\frac{dq}{dq_{20}}$$

q: canonical coordinate determined from (L)QRPA

$$\frac{dq}{dq_{20}} = \left\{\langle\phi(q_{20})|[\hat{Q}_{20}, \frac{1}{i}\hat{P}_i]|\phi(q_{20})\rangle\right\}^{-1}$$

P from LQRPA solution

$$q_{20} = \langle\phi(q_{20})|\hat{Q}_{20}|\phi(q_{20})\rangle$$

→ fission fragment distribution of r-process n-rich nuclei

Vibrational inertia: five-dimensional collective Hamiltonian (5DCH)

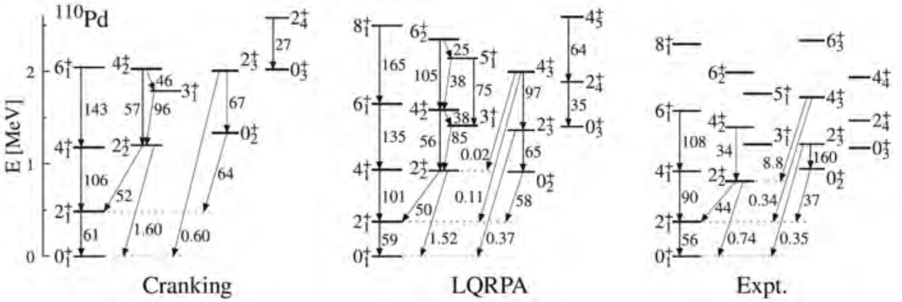
$$H_{\mathrm{coll}} = T_{\mathrm{vib}} + T_{\mathrm{rot}} + V(\beta,\gamma)$$

$$T_{\mathrm{vib}} = \frac{1}{2}D_{\beta\beta}(\beta,\gamma)\dot{\beta}^2 + D_{\beta\gamma}(\beta,\gamma)\dot{\beta}\dot{\gamma} + \frac{1}{2}D_{\gamma\gamma}(\beta,\gamma)\dot{\gamma}^2$$

$$T_{\mathrm{rot}} = \frac{1}{2}\sum_{k=1}^{3}\mathcal{J}_k(\beta,\gamma)(\omega_k^{\mathrm{rot}})^2$$

$D_{\beta\beta}$, $D_{\beta\gamma}$, $D_{\gamma\gamma}$: calculated from LQRPA(FAM)
J_1,J_2,J_3: Thouless-Valatin (FAM at zero energy)

Washiyama, NH, Nakatsukasa, Phys. Rev. C **109**, L051301 (2024)



→ systematic calculations of low-lying states including shape coexistence

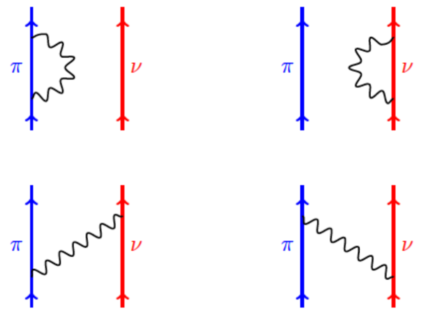
Quasiparticle vibration coupling for charge-exchange process in deformed systems

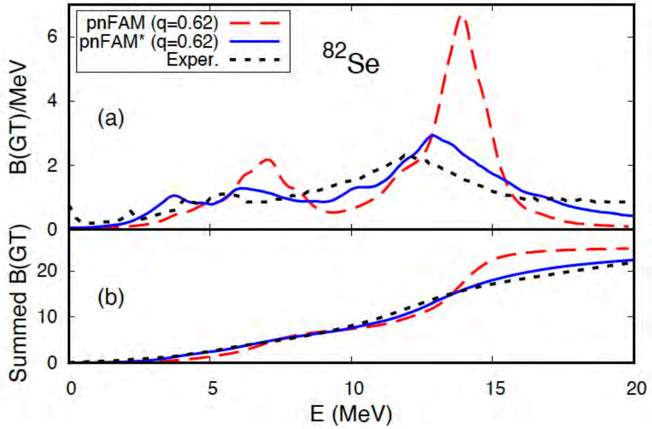
Liu, Engel, NH, Kortelainen, Phys. Rev. C **109**, 044308 (2024)

- isoscalar pn pairing was globally fitted to beta decay half-life within pnQRPA(pnFAM)
  Mustonen and Engel, Phys. Rev. C **93**, 014304 (2016)

- QRPA: includes coupling to two-quasiparticle states (Landau damping)
  no coupling to more complex configurations (multi-quasiparticles) (spreading width)
  (HO basis) no two-quasiparticle states coupling to continuum (escaping width)

- pnQRPA + quasiparticle-vibration coupling (pnFAM*)
  (without isoscalar pairing)





Isotope	β	$t_{1/2}^{\text{Expt.}}$ (s)	$t_{1/2}^{\text{pnFAM}}$ (s)	$t_{1/2}^{\text{pnFAM*}}$ (s)
^{78}Zn	0.11	1.47	15.8	3.57
^{168}Gd	0.31	3.03	280	17.7
^{152}Ce	0.30	1.40	79.8	6.05
^{156}Nd	0.32	5.49	406	32.5
^{164}Sm	0.33	1.42	66.9	9.96
^{154}Ce	0.30	0.30	18.3	3.04
^{112}Mo	−0.20	0.15	1.58	0.547
^{94}Kr	−0.23	0.21	3.12	1.29
^{112}Ru	−0.21	1.75	86.8	16.5
^{106}Mo	−0.21	8.73	106	19.1
^{96}Sr	−0.21	1.07	64.3	8.49

- qp vibration coupling increases allowed beta decay rate (same as isoscalar pairing)
- Optimization with qp vibration coupling may weaken the isoscalar paring

QRPA sum rule

NH, Kortelainen, Nazarewicz, Olsen, Phys. Rev. C **91**,044323 (2015)

Contour integration with multiple poles: summation over excited states

QRPA sum rule $$m_k(\hat{F}) = \sum_{i>0} \Omega_i^k |\langle i|\hat{F}|0\rangle|^2$$

FAM strength has first-order poles at QRPA excitation energies, with the transition strength as residues

$$S(\hat{F},\omega) = -\sum_{i>0}\left[\frac{|\langle i|\hat{F}|0\rangle|^2}{\Omega_i-\omega} + \frac{|\langle 0|\hat{F}|i\rangle|^2}{\Omega_i+\omega}\right]$$

$$m_k(\hat{F}) = \frac{1}{2\pi i}\oint_D \omega^k S(\hat{F},\omega)d\omega = \sum_{i>0} \Omega_i^k |\langle i|\hat{F}|0\rangle|^2$$

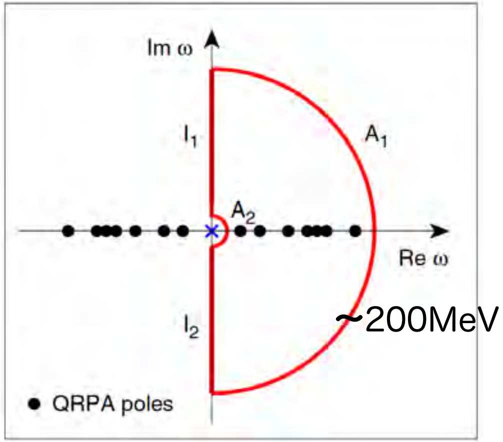
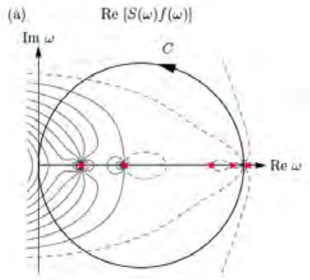


TABLE V. Sum rules (in MeV^k e^2 fm^4) for the isoscalar and i IEWSR pe EWSR ith the MQRPA and the FAM. The FAM calculations were performed by using $N_{A_1} = 301$, $N_{A_2} = 1$ nte

	$k=-4$	$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
MQRPA(ISM)	0.013 077	0.037 185	0.253 118	5.00 072	139.825	4200.82	131 368	4 342 358	157 906 069
FAM(ISM)	0.012 579	0.036 992	0.253 186	5.00 385	139.844	4199.44	131 277	4 336 644	157 058 272
MQRPA(IVM)	0.00 063 616	0.00 273 872	0.07 120 949	2.78 540	113.908	4735.03	199 525	8 527 358	370 625 216
FAM(IVM)	0.00 043 157	0.00 275 227	0.07 133 510	2.78 615	113.908	4734.30	199 510	8 524 830	368 643 941



beta decay: summation over all the states below Q-value

Mustonen et al., Phys. Rev. C **90**, 024308 (2014)

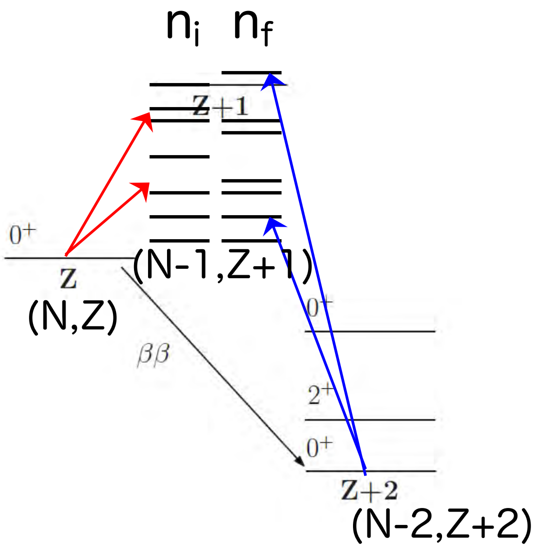
2νββ double-beta decay

two modes : neutrinoless mode (0νββ) and two-neutrino mode (2νββ)

NH, Engel, Phys. Rev. C **105**, 044314 (2022)

2νββ half-life $$[T^{2\nu}_{1/2}]^{-1} = G_{2\nu}(Q_{\beta\beta}, Z)|M^{2\nu}|^2$$

phase space factor nuclear matrix element (NME)



NME

$$M^{2\nu}_{\rm GT} = \sum_n \frac{\boxed{\langle 0^+_f | \sum_a \boldsymbol{\sigma}_a \tau^-_a | n\rangle} \cdot \boxed{\langle n | \sum_b \boldsymbol{\sigma}_b \tau^-_b | 0^+_i\rangle}}{E_n - (M_i + M_f)/2}$$

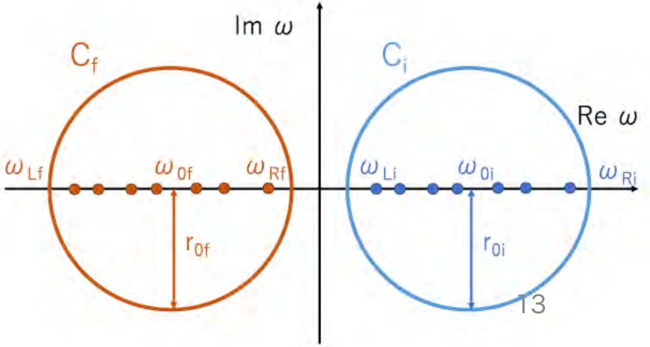
NME can be evaluated by combining two (pn)QRPA

FAM with two contour integrations

Two FAM amplitudes with Gamow-Teller external fields calculated from initial state (ω_i) and final state (ω_f) are combine

$$\mathcal{T}(\alpha; \omega_i, \hat{F}^{T_i}_{K_i}, \omega_f, \hat{F}^{T_f}_{K_f}) \equiv \sum_{pn} [\bar{Y}^{(f)}_{pn}(\omega_f, \hat{F}^{T_f}_{K_f}) \bar{X}^{(i)}_{pn}(\omega_i, \hat{F}^{T_i}_{K_i}) - \alpha \bar{X}^{(f)}_{pn}(\omega_f, \hat{F}^{T_f}_{K_f}) \bar{Y}^{(i)}_{pn}(\omega_i, \hat{F}^{T_i}_{K_i})]$$

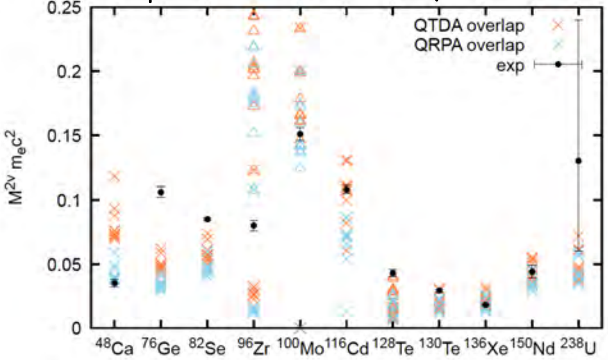
$$M^{2\nu}_{\rm GT} = -\sum_{K=-1}^{1} (-1)^K \left(\frac{1}{2\pi i}\right)^2 \oint_{C_i} d\omega_i \oint_{C_f} d\omega_f \mathcal{T}(\alpha, \omega_i, \hat{F}^{T_i}_{K_i}; \omega_f, \hat{F}^{T_f}_{K_f}) \frac{2}{\omega_i - \omega_f}$$



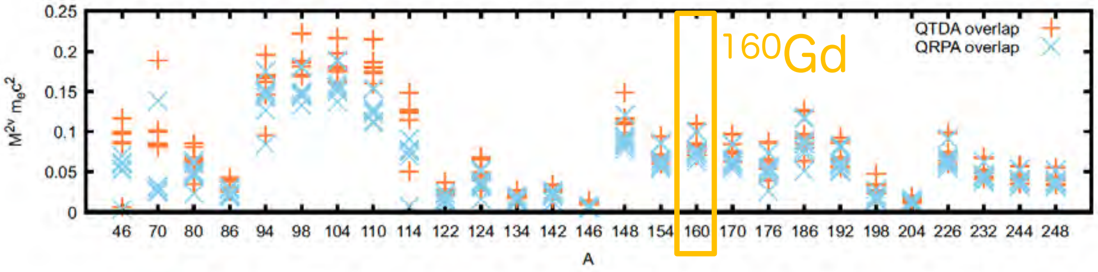
$2\nu\beta\beta$ double-beta decay

- First systematic calculation of $2\nu\beta\beta$ NME using DFT based QRPA NH and Engel, Phys. Rev. C **105**, 044314 (2022)
- Calculation with 10 EDF parameters globally fitted to (single) β-decay half-lives
  Mustonen and Engel, Phys. Rev. C **93**, 014304 (2016)

Comparison with data (11 nuclides)

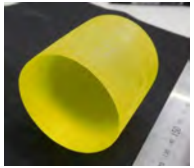


Prediction (27 nuclides)



PIKACHU experiment ($2\nu\beta\beta$ measurement of ^{160}Gd using GAGG crystal)

(Pure Inorganic scintillator experiment in KAmioka for CHallenging Underground sciences)



- Previous measurement in Ukraine: $2\nu\beta\beta$ half-life > 2.1×10^{19} y
  Danevich et al., Nucl. Phys. A **694**, 375 (2001)
- expected $2\nu\beta\beta$ half-life with previous NME： 6×10^{21} y
  Hirsch et al., Phys. Rev. C **66**, 015502 (2002)
- half-life expected from nuclear DFT-QRPA: 8×10^{20} y
- Recent publication: Omori et al., NIM-A **1082**, 171023 (2026)

PIKACHU Collaboration
Takashi Iida (Tsukuba) et al.

Summary

- Finite-amplitude method for QRPA in nuclear DFT
    - efficient method to solve QRPA problem based on linear response theory
    - applications to various phenomena within and beyond QRPA
        - giant resonances
        - zero mode (rotational/pairing rotational moment of inertia)
        - low-energy collective modes, collective inertia
        - sum rules
        - beta decay
        - two-neutrino double-beta decay

Collaborators

- Jyvaskyla: Markus Kortelainen
- FRIB/MSU : Witek Nazarewicz
- Ibaraki College, NIT: Tomohiro Oishi
- CiDER, Univ. of Osaka: Kouhei Washiyama
- UNC Chapel Hill: Qunqun Liu, Jon Engel